

THE J-FLOW ON TORIC MANIFOLDS

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ABSTRACT. We study the J-flow on the toric manifolds, through study the transition map between the moment maps induced by two Kähler metrics, which is a diffeomorphism between polytopes. This is similar to the work of Fang-Lai, under the assumption of Calabi symmetry, they study the monotone map between two intervals. We get a partial bound of the derivatives of transition map.

1. INTRODUCTION

In [4], Donaldson described the various situations where the diffeomorphism groups act on some spaces of maps between manifolds, these actions induce the moment maps, then various geometric flows arise as the gradient flow of the norm square of moment maps, and the J-flow is one of these. Moreover, in the study of the K-energy, Chen [2] introduce J-flow as the gradient flow of the J-functional. Let X be a Kähler manifold with a Kähler class $[\omega]$, the space of Kähler potentials is

$$\mathcal{H} = \{\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$$

Let α be a Kähler metric, the J-functional is defined on \mathcal{H} by

$$\mathcal{J}_{\alpha,\omega}(\varphi) = \int_0^1 \int_X \dot{\varphi}_t (\alpha \wedge \omega_{\varphi_t}^{n-1} - c\omega_{\varphi_t}^n) \frac{dt}{(n-1)!}$$

where $\{\varphi_t\}_{0 \leq t \leq 1}$ be any smooth path in \mathcal{H} from 0 to φ , and $c = \frac{\int \omega^{n-1} \wedge \alpha}{\int \omega^n}$, so $\mathcal{J}_{\alpha,\omega}(\varphi) = \mathcal{J}_{\alpha,\omega}(\varphi + a)$. The critical point φ of $\mathcal{J}_{\alpha,\omega}$ should satisfy the Donaldson's equation

$$(1.1) \quad c\omega_\varphi^n = \alpha \wedge \omega_\varphi^{n-1}$$

And the J-flow is

$$(1.2) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = nc - \frac{n\omega_\varphi^{n-1} \wedge \alpha}{\omega_\varphi^n} \\ \varphi|_{t=0} = \varphi_0 \end{cases}$$

Chen [3] showed the long time existence of J-flow and the convergence to solution of (1.1) when α have non-negative bisectional curvature. Then in the work of Song-Weinkove [12], by a more delicate estimate based on the previous work of Weinkove [17, 18], a necessary and sufficient condition for convergence is found,

J-flow (1.2) converges to the solution of (1.1) if and only if there exists a metric $\omega' \in [\omega]$ such that the $(n-1, n-1)$ type form

$$(1.3) \quad nc\omega'^{n-1} - (n-1)\omega'^{n-2} \wedge \alpha > 0$$

This condition almost assume that there exists a subsolution of (1.1). However it is hard to check for concrete example. In particular, it is hard to see from this condition that whether the convergence depends on the choice of α in its class $[\alpha]$, in other words, if the solvability of (1.1) only depends on the class $[\omega]$ and $[\alpha]$.

In [10], Lejmi and Székelyhidi study the solvability of (1.1) from the view of geometric stability, as the problem of the existence of cscK metrics, that is conjectured be equivalent to the K-stability of manifold. Let L be a line bundle on X , $[\omega] = c_1(L)$, for a test-configuration χ for (X, L) , they define an invariant $F_\alpha(\chi)$ which is similar to the Donaldson-Futaki invariant when we study the Kähler-Einstein metrics with conical singularity. It is proved that if (1.1) have a solution then $F_\alpha(\chi) > 0$ for any test-configuration χ with positive norm, this is corresponding to the result in [14] which is for the cscK metrics. In particular, when χ is coming from the deformation to the normal cone of a subvariety (see [11]), the corresponding condition $F_\alpha(\chi) > 0$ is

For all p -dimensional subvariety V of X , where $p = 1, 2, \dots, n-1$, we have

$$(1.4) \quad nc \int_V \frac{\omega^p}{p!} > \int_V \frac{\omega^{p-1}}{(p-1)!} \wedge \alpha$$

Obviously these conditions only depend on the classes. They also conjecture that (1.4) would be sufficient condition for (1.1) have solution. Moreover, (1.4) can be derived directly, suppose (1.1) have solution φ , for smooth point $x \in V$, choose coordinate z^i such that $V = \{z^{p+1} = \dots = z^n = 0\}$ near x , and $\omega_{\varphi, i\bar{j}} = \delta_{i\bar{j}}$ at x , since

$$p\omega_{\varphi, V}^{p-1} \wedge \alpha_V = \text{tr}_{\omega_{\varphi, V}} \alpha_V \omega_{\varphi, V}^p$$

where α_V is the restriction of α on V . The trace $\text{tr}_{\omega_{\varphi, V}} \alpha_V = \sum_{i \leq p} \alpha_{i\bar{i}} < \sum \alpha_{i\bar{i}} = nc$, so $nc\omega_{\varphi, V}^p - p\omega_{\varphi, V}^{p-1} \wedge \alpha_V > 0$ at x , then integrate it over V_{reg} is (1.4). In the same way, we see $nc\omega_\varphi - \alpha > 0$, so on the class level $nc[\omega] - [\alpha] > 0$. When $n = 2$, by (1.3) this is a necessary and sufficient condition for solving (1.1), but when $n > 2$ it is not sufficient, see counter-example in [10].

When $n = 2$, Donaldson [4] noted that the above condition $nc[\omega] - [\alpha] > 0$ is satisfied for all Kähler classes if there not exist curves with negative self-intersection, and conjectured that if this condition is violated, the flow (1.2) will blow up over these curves.

In [12], they confirm the above conjecture in a partial sense. More recently, in [8, 13], they consider the situations where $nc[\omega] - [\alpha] \geq 0$ or α

degenerate along a divisor, it is proved that the flow will converge outside a union set of curves. The argument heavily depends on the fact that when $n = 2$ (1.1) can be transformed to the Monge-Ampère equation, and the latter has continuous solution in the degenerate case, due to the work of Eyssidieux-Guedj-Zeriahi.

For the concrete example, Fang and Lai [7] study the long time behavior of J-flow on the projective bundles $X_{m,n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$, $X_{0,n}$ is the \mathbb{P}^{n+1} blow-up one point. Under the Calabi symmetry assumption, the flow can be described by a time-dependent monotone map between two intervals, through solve the static equation (ODE), they see the flow always converges to a Kähler current and on its smooth region satisfies (1.1).

In this paper, we study J-flow on the toric manifolds and assume the metrics are invariant under torus action. We expect to find more verifiable conditions which can ensure the flow converges, maybe (1.4) for the invariant subvarieties or some combinatorial conditions for polytopes. If the flow does not converge, we also want to understand its asymptotic behavior.

After the symmetry reduction, (1.2) can be defined on \mathbb{R}^n by

$$\frac{\partial \phi_t}{\partial t} = nc - \sum_{i,j} f_{ij} \phi_t^{ij}$$

where ϕ_t and f are potentials for ω_{φ_t} and α respectively, they conform the asymptotic behavior at infinity assigned by the polytope \mathcal{P} and \mathcal{Q} . Through the Legendre transform of ϕ_t , this nonlinear equation be transformed to a quasilinear one which is defined on \mathcal{P} ,

$$(1.5) \quad \frac{\partial u_t}{\partial t} = \sum_{i,j} f_{ij} (\nabla u_t) u_{ij} - nc$$

u_t is the Legendre transform of ϕ_t , satisfies Guillemin's boundary condition, so ∇u_t blow up near the boundary, since $[f_{ij}]$ degenerate at infinity, so (1.5) is degenerate on the boundary, the RHS is even not defined on the boundary. Moreover, when the flow does not converge, by the example 4.7, we see ∇u_t may blow up in a whole domain located in \mathcal{P} as $t \rightarrow \infty$.

We turn to study the transition map $U_t = \nabla f \circ \nabla u_t$ between the moment maps induced by ω_{φ_t} and α , it is a diffeomorphism between polytopes and map the face to face. The price is U_t satisfies a degenerate parabolic system (4.3). The static map satisfies

$$\text{tr} DU \equiv nc$$

In the paper we just get a partial bound on DU_t , we conjecture that DU_t is bounded uniformly w.r.t. time, and U_t will converge to a limit map U_∞ even if the origin J-flow does not converge, but in this case U_∞ must degenerate on some domain in the sense $\det DU_\infty = 0$, since if $\det DU > \delta$ uniformly imply the flow converges. This degeneracy may violate the condition (3.6).

We also want to know if J-flow minimizes the functional, namely if

$$E_\alpha(\omega_{\varphi_t}) \rightarrow \inf_{\omega \in [\omega]} E_\alpha(\omega)$$

where

$$E_\alpha(\omega) = \frac{1}{2} \int_X (tr_\omega \alpha)^2 \frac{\omega^n}{n!} = \frac{1}{2} \int_{\mathcal{P}} (tr DU)^2 dy$$

If it does, $tr DU_\infty$ may correspond to the worst test-configuration which be discussed in [10].

2. TORIC MANIFOLDS AND POTENTIALS

We review the Kähler structure on toric manifolds in detail, since we need the coordinate charts which include the invariant divisors, the logarithmic coordinate defined on the dense open set push these divisors to infinity. These coordinates also been introduced in [6], here we give a basis-free formulation. More details of toric variety see [9].

Let (\mathbb{C}, \times) , $(\mathbb{R}_\geq, \times)$ be semi-groups, \mathbb{R}_\geq is the set of non-negative real number. N is a lattice with rank n , its dual lattice is M , given a Delzant polytope \mathcal{P} in $M_R = M \otimes_{\mathbb{Z}} \mathbb{R}$, and suppose $\{u_i\} \subset N$ be the prime inward normal vectors of the facets of \mathcal{P} , then \mathcal{P} is

$$(2.1) \quad \mathcal{P} = \{y \in M_R \mid d_i(y) = \langle u_i, y \rangle + b_i \geq 0, \text{ for all } i\}$$

it induces a fan Σ (a collection of cones) in $N_R = N \otimes_{\mathbb{Z}} \mathbb{R}$. For a vertex q of \mathcal{P} , it corresponds a n -dimensional cone σ_q in Σ ,

$$\sigma_q = \{u \in N_R \mid \langle u, q \rangle = \min_{\mathcal{P}} \langle u, \cdot \rangle\}$$

its dual cone $\sigma_q^\vee = \{y \in M_R \mid \langle \sigma_q, y \rangle \geq 0\}$ is generated by $\mathcal{P} - q$, and the semi-group $\sigma_q^\vee \cap M$ is finitely generated, so we can construct a finitely generated algebra $\mathbb{C}[\sigma_q^\vee \cap M] = \{\sum_v a_v \chi^v \mid v \in \sigma_q^\vee \cap M, a_v \in \mathbb{C}\}$ with multiplication $\chi^v \cdot \chi^{v'} = \chi^{v+v'}$, it defines an affine open set U_q

$$U_q = \text{Spm } \mathbb{C}[\sigma_q^\vee \cap M] = \{\varphi \mid \varphi : \sigma_q^\vee \cap M \rightarrow \mathbb{C}\}$$

where φ is a homomorphism between semi-groups. Let $e_1^q, \dots, e_n^q \in M$ be the prime vectors rooted at q and along the edges of \mathcal{P} , it is a basis of M due to Delzant's conditions, and generates $\sigma_q^\vee \cap M$. Assume $\varphi(e_i^q) = z_i^q$, then $U_q \cong \mathbb{C}^n = \{(z_1^q, \dots, z_n^q)\}$. U_q include the dense open subset $U_0 = \text{Spm } \mathbb{C}[M] = \{\varphi : M \rightarrow \mathbb{C}^*\} \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$. $\{U_q\}$ glues with each other to form a toric manifold X_Σ with a torus $\text{Spm } \mathbb{C}[M]$ action.

The subset $U_q^\geq = \{\varphi : \sigma_q^\vee \cap M \rightarrow \mathbb{R}_\geq\} \cong \mathbb{R}_\geq^n$ is called the non-negative part of U_q , $\pi_q : U_q \rightarrow U_q^\geq$ is defined by $\varphi \mapsto |\varphi|^2$. In the same way we have $\pi_0 : U_0 \rightarrow U_0^\geq = \{\varphi : M \rightarrow \mathbb{R}^+\} \cong N \otimes_{\mathbb{Z}} \mathbb{R}^+$. Since $(\mathbb{R}^+, \times) \cong (\mathbb{R}, +)$ by $a \mapsto \log a$, we identify $N \otimes_{\mathbb{Z}} \mathbb{R}^+$ with $N \otimes_{\mathbb{Z}} \mathbb{R} = N_R$. $\{U_q^\geq\}$ glues with each other to form a closed subset $X_\Sigma^\geq \subset X_\Sigma$, and $\{\pi_q\}$ glues to a continuous map $\pi : X_\Sigma \rightarrow X_\Sigma^\geq$.

$$\begin{array}{ccccccc}
& & U_0 & \hookrightarrow & U_q & \hookrightarrow & X_\Sigma \\
& & \downarrow \pi_0 & & \downarrow \pi_q & & \downarrow \pi \\
M_R & \xleftarrow{d\phi} & N_R & \xleftarrow{\cong} & U_0^\geq & \hookrightarrow & U_q^\geq \hookrightarrow X_\Sigma^\geq
\end{array}$$

FIGURE 2.1.

Let u be a symplectic potential, that is a convex function defined on $\bar{\mathcal{P}}$ and satisfies

- restrict u in the interior of a face, it is smooth and strictly convex.
- Guillemin boundary condition, $u = \sum d_i(y) \log d_i(y) + v$, $v \in \mathcal{C}^\infty(\bar{\mathcal{P}})$.

Then u induces an invariant Kähler metric ω_u in following way, let ϕ be the Legendre transform of u , which is defined on N_R , for $x \in N_R$,

$$\phi(x) = \langle x, y \rangle - u(y), \quad x = du(y), \quad y = d\phi(x)$$

Take a vertex q , let $\phi_q = \phi - \langle \cdot, q \rangle$, ϕ_q can be defined on U_0^\geq through $U_0^\geq \cong N \otimes_{\mathbb{Z}} \mathbb{R}^+ \cong N_R$. Since $U_0^\geq \subset U_q^\geq$, ϕ_q can be extended smoothly on U_q^\geq due to the Guillemin's boundary condition, denote the extension is $\bar{\phi}_q$. Let $\Phi_q = \bar{\phi}_q \circ \pi_q$, then $\omega_q = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Phi_q > 0$ on U_q due to the convexity of u . Since $\phi_q - \phi_{q'} = \langle \cdot, q' - q \rangle$, let χ_q be the pullback of $\langle \cdot, q \rangle$ by the composed map $U_0 \rightarrow U_0^\geq \rightarrow N_R$, so on U_0 , we have $\Phi_q - \Phi_{q'} = \chi_{q'-q}$. Take a basis of M , χ_q is $(z_1, \dots, z_n) \mapsto q_1 \log |z_1|^2 + \dots + q_n \log |z_n|^2$, so $\partial \bar{\partial} \chi_q = 0$. Hence $\omega_q = \omega_{q'}$ on U_0 , so is on $U_q \cap U_{q'}$ since U_0 is dense, so $\{\omega_q\}$ defines ω_u . In another way, let Φ be the pullback of ϕ by $U_0 \rightarrow U_0^\geq \rightarrow N_R$, then $\omega_u = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Phi$ on U_0 , and the extension of $\Phi - \chi_q$ to U_q is Φ_q .

The map $U_0^\geq \rightarrow N_R \xrightarrow{d\phi} M_R$ can be extended to X_Σ^\geq as a homeomorphism onto $\bar{\mathcal{P}}$, compose it with π is the moment map μ of (X_Σ, ω_u) with respect to the torus action.

The Kähler class $[\omega_u] = \sum b_i [D_i]$, D_i is the invariant divisor corresponding to u_i .

Now we take a basis of M and write down the above things explicitly, let $\{e_i^q\}$ be the basis mentioned above, and the dual basis of N is $\{v_q^i\}$, it is the prime generators of the cone σ_q .

Then U_q have coordinates z_i^q , $U_q^\geq \cong \mathbb{R}_{\geq}^n$ have coordinates $a_i^q \geq 0$, and $U_0^\geq = \{(a_i^q) \mid a_i^q > 0\}$, N_R have coordinates x_i^q , M_R have coordinates y_q^i . The map $\pi_q : U_q \rightarrow U_q^\geq$ is $(z_i^q) \mapsto (|z_i^q|^2)$, and the identification $U_0^\geq \rightarrow N_R$ is $(a_i^q) \mapsto (\log a_i^q)$.

In the following we omit the index q of variables for simplicity, x_i , a_i always means x_i^q , a_i^q , etc.

Let

$$u_q(y^1, \dots, y^n) = u(q + y^1 e_1^q + \dots + y^n e_n^q), \quad y^i \geq 0$$

Then the Legendre transform of u_q is $\phi_q(\sum x_i v_q^i)$, $x_i = \partial u_q / \partial y^i$, denote it as $\phi_q(x_i) \triangleq \phi_q(x_1, \dots, x_n)$ for short. Since the map $U_0 \rightarrow U_0^\geq \rightarrow N_R$ is $(z_i) \mapsto \sum \log |z_i|^2 v_q^i$, so $\Phi_q(z_i) = \phi_q(\log |z_i|^2)$, it can be extended smoothly on \mathbb{C}^n as a Kähler potential, and $\bar{\phi}_q(a_i) = \phi_q(\log a_i)$ can be extended smoothly on \mathbb{R}_{\geq}^n . We have three functions $\Phi_q(z_i)$, $(z_i) \in \mathbb{C}^n$, $\bar{\phi}_q(a_i)$, $(a_i) \in \mathbb{R}_{\geq}^n$, $\phi_q(x_i)$, $(x_i) \in \mathbb{R}^n$ and satisfy

$$\Phi_q(z_i) = \bar{\phi}_q(a_i) = \phi_q(x_i), \quad a_i = |z_i|^2, \quad x_i = \log a_i$$

For the convexity of these functions, on \mathbb{C}^n

$$(2.2) \quad \left[\frac{\partial^2 \Phi_q}{\partial z_i \partial \bar{z}_j} \right] = \left[\delta_{ij} \frac{\partial \bar{\phi}_q}{\partial a_i} + \bar{z}_i z_j \frac{\partial^2 \bar{\phi}_q}{\partial a_i \partial a_j} \right] > 0$$

On $(\mathbb{C}^*)^n$

$$(2.3) \quad \left[\frac{\partial^2 \Phi_q}{\partial z_i \partial \bar{z}_j} \right] = \left[\frac{1}{z_i \bar{z}_j} \frac{\partial^2 \phi_q}{\partial x_i \partial x_j} \right] > 0$$

ϕ_q is strictly convex on \mathbb{R}^n . For $\bar{\phi}_q$, note that when $z_i, z_j \neq 0$

$$(2.4) \quad \delta_{ij} \frac{\partial \bar{\phi}_q}{\partial a_i} + \bar{z}_i z_j \frac{\partial^2 \bar{\phi}_q}{\partial a_i \partial a_j} = \left(\delta_{ij} a_i \frac{\partial \bar{\phi}_q}{\partial a_i} + a_i a_j \frac{\partial^2 \bar{\phi}_q}{\partial a_i \partial a_j} \right) \frac{1}{z_i \bar{z}_j}$$

we can see the RHS of (2.2) is positive definite if and only if $\bar{\phi}_q$ satisfies

- $\frac{\partial \bar{\phi}_q}{\partial a_i} > 0$ when $a_i = 0$
- for any $\Lambda \subseteq \{1, \dots, n\}$, it can be empty, on coordinate plane $\{a_i > 0, i \in \Lambda, a_i = 0, i \notin \Lambda\}$,

$$\left[\delta_{ij} a_i \frac{\partial \bar{\phi}_q}{\partial a_i} + a_i a_j \frac{\partial^2 \bar{\phi}_q}{\partial a_i \partial a_j} \right]_{i,j \in \Lambda} > 0$$

$\bar{\phi}_q$ and u_q transform to each other in a way similar to the Legendre transform. For example, on coordinate plane $\{y \mid y^n = 0, y^i \neq 0, i < n\}$ corresponding to the facet of polytope,

$$u_q(y^i, 0) = \sum_{i < n} y^i \log a_i - \bar{\phi}_q(a_i, 0), \quad y^i = a_i \frac{\partial \bar{\phi}_q}{\partial a_i}(a_k, 0), \quad \text{for } i < n$$

So $\bar{\phi}_q|_{a_n=0}$ is determinate by $u_q|_{y^n=0}$.

The moment map¹ $X_{\Sigma}^\geq \rightarrow \bar{\mathcal{P}}$, restrict on $U_q^\geq \cong \mathbb{R}_{\geq}^n$ is

$$(2.5) \quad (a_i) \mapsto q + a_1 \frac{\partial \bar{\phi}_q}{\partial a_1} e_1^q + \dots + a_n \frac{\partial \bar{\phi}_q}{\partial a_n} e_n^q$$

¹We call it as moment map just for convenient.

it is smooth since $\bar{\phi}_q$ is smooth on \mathbb{R}_{\geq}^n . The inverse map is

$$(2.6) \quad q + y^1 e_1^q + \cdots + y^n e_n^q \mapsto \left(\exp \frac{\partial u_q}{\partial y^i} \right)$$

Since u satisfies the Guillemin's boundary condition, $u_q = \sum y^i \log y^i + v(y)$, v is smooth to the boundary, so $\exp \frac{\partial u_q}{\partial y^i} = y^i \exp(1 + \frac{\partial v}{\partial y^i})$ is smooth. Hence $X_{\Sigma}^{\geq} \rightarrow \bar{\mathcal{P}}$ is actually a diffeomorphism.

On U_0^{\geq} , compose (2.5) and its inverse (2.6), we have

$$\frac{\partial u_q}{\partial y^i} \left(a_1 \frac{\partial \bar{\phi}_q}{\partial a_1}, \dots, a_n \frac{\partial \bar{\phi}_q}{\partial a_n} \right) = \log a_i$$

Take derivative w.r.t. a_j ,

$$\sum_k \frac{\partial^2 u_q}{\partial y^i \partial y^k} \left(\delta_{kj} a_j \frac{\partial \bar{\phi}_q}{\partial a_j} + a_k a_j \frac{\partial^2 \bar{\phi}_q}{\partial a_k \partial a_j} \right) = \delta_{ij}$$

so on U_0^{\geq} ,

$$(2.7) \quad \left[\delta_{ij} a_i \frac{\partial \bar{\phi}_q}{\partial a_i} + a_i a_j \frac{\partial^2 \bar{\phi}_q}{\partial a_i \partial a_j} \right]^{-1} (a_k) = \left[\frac{\partial^2 u_q}{\partial y^i \partial y^j} \right] (a_k \frac{\partial \bar{\phi}_q}{\partial a_k})$$

with (2.2) and (2.4), $a_i = |z_i|^2$, we have

$$(2.8) \quad \left[\frac{\partial^2 \Phi_q}{\partial z_i \partial \bar{z}_j} \right]^{-1} (z_i) = \left[\bar{z}_i z_j \frac{\partial^2 u_q}{\partial y^i \partial y^j} \right] (a_i \frac{\partial \bar{\phi}_q}{\partial a_i})$$

On the coordinate plane we have similar formula, for example on $\{z_n = z_{n-1} = 0, z_\alpha \neq 0\}$,

$$(2.9) \quad \left[\frac{\partial^2 \Phi_q}{\partial z_i \partial \bar{z}_j} \right]^{-1} = \begin{bmatrix} \left(\bar{z}_i z_j \frac{\partial^2 u_q}{\partial y^i \partial y^j} \right)_{i,j < n-1} & 0 & 0 \\ 0 & (\frac{\partial \bar{\phi}_q}{\partial a_{n-1}})^{-1} & 0 \\ 0 & 0 & (\frac{\partial \bar{\phi}_q}{\partial a_n})^{-1} \end{bmatrix}$$

We know $[u_{q,ij}]$ is singular on the boundary of polytope, however its inverse can be extended smoothly to the boundary.

Proposition 2.1. *The inverse of $[u_{q,ij}]$ can be extended smoothly on $\bar{\mathcal{P}}$.*

Proof. From (2.7) and (2.6) we know the inverse

$$(2.10) \quad [u_q^{ij}](y^k) = \left[\delta_{ij} a_i \frac{\partial \bar{\phi}_q}{\partial a_i} + a_i a_j \frac{\partial^2 \bar{\phi}_q}{\partial a_i \partial a_j} \right] \left(\exp \frac{\partial u_q}{\partial y^k} \right)$$

Obviously it is can be extended to the boundary, moreover $[u_q^{ij}]$ is semi-positive definite on the face, and positive definite in the tangent space of face. \square

3. TRANSITION BETWEEN MOMENT MAPS

Let \mathcal{P} and \mathcal{Q} be defined by (2.1) with different collection of b_i , they have similar shape, for every face of \mathcal{P} there is a corresponding face of \mathcal{Q} parallel to it. Then induce same fan Σ , so same toric manifold X_Σ . Let $[\omega]$ and $[\alpha]$ be the corresponding Kähler class. The invariant Kähler metric ω induces a moment map $\mu_\omega : X_\Sigma \rightarrow \bar{\mathcal{P}}$, and α induces another one $\mu_\alpha : X_\Sigma \rightarrow \bar{\mathcal{Q}}$. It is well known that μ_ω establishes a one-to-one correspondence between the real torus orbits and the points of $\bar{\mathcal{P}}$, so there is a transition map $U : \bar{\mathcal{P}} \rightarrow \bar{\mathcal{Q}}$ such that $U \circ \mu_\omega = \mu_\alpha$, it is a homeomorphism. We next check it is actually a diffeomorphism.

Let $u(y)$ and $g(y)$ be the symplectic potentials of ω and α , they are defined on $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ respectively. Take a vertex q of \mathcal{P} , the corresponding vertex of \mathcal{Q} is q' . It corresponds an affine open set $U_q = U_{q'}$, we have coordinates on it. Without loss of generality², we assume $q = q' = 0$, so $\phi = \phi_q$, $u = u_q$. In the following, we omit the index q, q' of potentials, and denote the potentials $\Phi, \bar{\phi}, \phi$ for ω , F, \bar{f}, f for α , such that $\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Phi = \frac{\sqrt{-1}}{2\pi} \Phi_{i\bar{j}} dz^i \wedge d\bar{z}^j$, $\alpha = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} F = \frac{\sqrt{-1}}{2\pi} F_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

Assume

$$U(q + y^1 e_1^q + \cdots + y^n e_n^q) = q' + \sum_i U^i(y) e_i^q$$

Note that μ_ω is the composition of π with $X_\Sigma^\geq \rightarrow \bar{\mathcal{P}}$, π is independent of metrics, then compose (2.5) for \bar{f} and (2.6) for u , we have

$$(3.1) \quad U^i(y) = \frac{\partial \bar{f}}{\partial a_i} \left(\exp \frac{\partial u}{\partial y^1}, \cdots, \exp \frac{\partial u}{\partial y^n} \right) \exp \frac{\partial u}{\partial y^i}$$

Since $\exp \frac{\partial u}{\partial y^i} = y^i \exp(1 + \frac{\partial v}{\partial y^i})$ is smooth due to the Guillemin's boundary condition, so U is a diffeomorphism.

When $y^k > 0$ for all k , namely in the interior of $\bar{\mathcal{P}}$, by $\bar{f}(a_i) = f(\log a_i)$,

$$U^i(y) = \frac{\partial f}{\partial x_i} \left(\frac{\partial u}{\partial y^1}, \cdots, \frac{\partial u}{\partial y^n} \right)$$

namely $(U^i) = \nabla f(\nabla u)$, since $\nabla f : \mathbb{R}^n \rightarrow \mathcal{Q}$ is a diffeomorphism and the inverse is ∇g , so $\nabla u = \nabla g(U)$. Change the order of derivatives, $u_{ij} = u_{ji}$, we see U must satisfy a compatible condition,

$$(3.2) \quad \sum_k g^{ik}(U) \frac{\partial U^j}{\partial y^k} = \sum_k g^{jk}(U) \frac{\partial U^i}{\partial y^k}$$

by Proposition 2.1, g^{ik} is smooth on $\bar{\mathcal{Q}}$, so above identity is actually valid on $\bar{\mathcal{P}}$.

²We can translate the polytope such that $q = 0$.

Remark 3.1. The compatible condition can be described in a natural way, for a point $y \in \mathcal{P}$, we define a metric on $T_y M_R = M_R$ by $\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle = g_{ij}(U(y))$, then (3.2) says $DU|_y : M_R \rightarrow M_R$ is self-dual and positive, positive is because $\langle DU \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle = \frac{\partial U^k}{\partial y^i} g_{kj}(U) = f_{kl}(\nabla u) u_{li} g_{kj}(U) = u_{ji}$ is positive definite. If $y \in \partial P$, we define the metric on the tangent space of face, and restrict $DU|_y$ on the tangent space is self-dual and positive.

For metric ω and α , we defined a linear transform A on $T^{1,0}X$ by

$$\langle A\partial_i, \partial_j \rangle_\omega = \langle \partial_i, \partial_j \rangle_\alpha$$

where $\langle \partial_i, \partial_j \rangle_\omega = \Phi_{i\bar{j}}$. Suppose $A\partial_i = A_i^k \partial_k$, then $A_i^k = F_{i\bar{j}} \Phi^{k\bar{j}}$. A is self-dual respect to ω and α , namely $\langle A\partial_i, \partial_j \rangle_\omega = \langle \partial_i, A\partial_j \rangle_\omega$, $\langle A\partial_i, \partial_j \rangle_\alpha = \langle \partial_i, A\partial_j \rangle_\alpha$, and positive $\langle Ax, x \rangle_\omega = \langle x, x \rangle_\alpha > 0$ for $0 \neq x \in T^{1,0}M$.

On $U_0 \cong (\mathbb{C}^*)^n$, by (2.3) and (2.8),

$$(3.3) \quad A_i^k = F_{i\bar{j}} \Phi^{k\bar{j}} = \frac{1}{z_i \bar{z}_j} f_{ij} z_k \bar{z}_j u_{kj} = \frac{z_k}{z_i} \frac{\partial U^i}{\partial y^k}$$

The characteristic polynomial of $A : T^{1,0}M \rightarrow T^{1,0}M$

$$\det(id + tA) = \det[\delta_i^k + tA_i^k] = \det\left[\frac{z_k}{z_i}(\delta_i^k + t \frac{\partial U^i}{\partial y^k})\right] = \det(id + tDU)|_{\mu_\omega}$$

This identity holds on X by continuation.

Proposition 3.2. *$A : T^{1,0}M \rightarrow T^{1,0}M$ with $DU : M_R \rightarrow M_R$ have the same characteristic polynomial. In particular, the eigenvalues of DU are positive.*

Note that

$$\det(id + tA) = \frac{(\omega + t\alpha)^n}{\omega^n}$$

so we have

$$\frac{(\omega + t\alpha)^n}{\omega^n} = \det(id + tDU)|_{\mu_\omega}$$

In particular,

$$(3.4) \quad tr_\omega \alpha = \frac{n\omega^{n-1} \wedge \alpha}{\omega^n} = tr A = tr DU|_{\mu_\omega}, \quad \frac{\alpha^n}{\omega^n} = \det A = \det DU|_{\mu_\omega}$$

Donaldson's equation (1.1) is $tr DU \equiv nc$, and U is subject to (3.2).

Moreover, when we restrict metrics on the invariant subvariety, we have

Proposition 3.3. *Assume F is a p -dimensional face of \mathcal{P} , V is the invariant subvariety corresponding to F , ω_V is the restriction of ω on V , $U|_F : F \rightarrow F'$ is the restriction of U , F' is the corresponding face of \mathcal{Q} , then on V*

$$\frac{(\omega_V + t\alpha_V)^p}{\omega_V^p} = \det(id + tD(U|_F))|_{\mu_\omega}$$

where μ_ω map V onto \bar{F} , $D(U|_F) = DU|_{TF}$ is the tangent map of $U|_F$, since $TF = TF'$, it is a linear transform of TF .

Proof. Just note that $\mu_\omega|_V : V \rightarrow \bar{F}$ is the moment map of (V, ω_V) w.r.t. the real torus action. \square

In particular, on V

$$(3.5) \quad \alpha_V \wedge \frac{\omega_V^{p-1}}{(p-1)!} = (tr DU|_{TF})|_{\mu_\omega} \frac{\omega_V^p}{p!}$$

Note that the push-out measure of $\frac{\omega_V^p}{p!}$ by $\mu_\omega|_V$ is σ_F , the canonical measure on F . First we have the Lebesgue measure Ω on M_R induced by the lattice M , and let $\{u_i\}_{i=1}^{n-p}$ be the generators that vanished along TF , then $\sigma_F = (\iota_{v_1} \cdots \iota_{v_{n-p}} \Omega)|_{TF}$, where $\{v_i\} \subset M_R$ such that $\langle u_i, v_j \rangle = \delta_{ij}$. Integrate (3.5), we have

$$\int_V \alpha_V \wedge \frac{\omega_V^{p-1}}{(p-1)!} = \int_F tr DU|_{TF} d\sigma_F$$

In particular, take $F = \mathcal{P}$

$$\int_X \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_{\mathcal{P}} \sum_i \frac{\partial U^i}{\partial y^i} dy$$

Then the condition (1.4) for the invariant subvarieties is for any p -dimensional face F of \mathcal{P}

$$(3.6) \quad \frac{1}{vol(F)} \int_F tr DU|_{TF} d\sigma_F < nc = \frac{1}{vol(\mathcal{P})} \int_{\mathcal{P}} tr DU dy$$

If (1.4) have a solution, we can derived this directly. The solution induces a transition map U such that $tr DU \equiv nc$. Since on F , TF is invariant under DU , it induces $DU|_{M_R/TF} : M_R/TF \rightarrow M_R/TF$, and its trace must be positive, so

$$tr DU = tr DU|_{TF} + tr DU|_{M_R/TF} > tr DU|_{TF}$$

integrate this inequality over F , we get (3.6).

4. THE FLOW OF TRANSITION MAPS

Suppose ω_{φ_t} be the solution of (1.2), if the initial data ω, φ_0 are invariant then φ_t is always invariant. Assume ω and α has potential ϕ_0 and f defined on N_R respectively, then ω_{φ_t} has potential $\phi_t = \phi_0 + \varphi_t$, denote the Legendre transform of ϕ_t is u_t defined on $\bar{\mathcal{P}}$, namely the symplectic potential of ω_{φ_t} . The transition map from $\bar{\mathcal{P}}$ to $\bar{\mathcal{Q}}$ induced by ω_{φ_t} and α is U_t .

Fix $y \in \mathcal{P}$, assume $x_t \in N_R$ such that

$$d\phi_t(x_t) = y$$

then

$$\begin{aligned} u_t(y) &= \langle x_t, y \rangle - \phi_t(x_t) \\ \frac{\partial u_t}{\partial t}(y) &= \left\langle \frac{dx_t}{dt}, y \right\rangle - \frac{\partial \phi_t}{\partial t}(x_t) - \left\langle d\phi_t, \frac{dx_t}{dt} \right\rangle = -\frac{\partial \phi_t}{\partial t}(x_t) = -\frac{\partial \varphi_t}{\partial t}(x_t) \end{aligned}$$

by this and (1.2), (3.4), we have

$$(4.1) \quad \frac{\partial u_t}{\partial t} = \sum f_{ij}(\nabla u_t) u_{ij} - nc, \text{ on } \mathcal{P}$$

and

$$\frac{\partial u_t}{\partial t} = \sum_i \frac{\partial U^i}{\partial y^i} - nc, \text{ on } \bar{\mathcal{P}}$$

since $\frac{\partial u_t}{\partial t}$ is smooth on $\bar{\mathcal{P}}$.

Next we consider the evolve equation of U_t , on \mathcal{P}

$$\begin{aligned} \frac{\partial U^i}{\partial t} &= \frac{\partial}{\partial t}(f_i(\nabla u_t)) = \sum_j f_{ij}(\nabla u_t) \frac{\partial}{\partial y^j} \left(\frac{\partial u_t}{\partial t} \right) \\ &= \sum_{j,k} f_{ij}(\nabla u_t) \frac{\partial^2 U^k}{\partial y^j \partial y^k} = \sum_{j,k} g^{ij}(U) \frac{\partial^2 U^k}{\partial y^j \partial y^k} \end{aligned}$$

Recall that $[g^{ij}] = [\partial_i \partial_j g]^{-1}$ is smooth on $\bar{\mathcal{Q}}$, so actually on the whole $\bar{\mathcal{P}}$, we have

$$(4.2) \quad \frac{\partial U^i}{\partial t} = \sum_{j,k} g^{ij}(U) \frac{\partial^2 U^k}{\partial y^j \partial y^k}$$

From (2.10) we know exactly how $[g^{ij}]$ degenerate at boundary, when $U \in F'$ the vector $\left(\sum_{j,k} g^{ij}(U) \frac{\partial^2 U^k}{\partial y^j \partial y^k} \right)^i$ is located in the tangent space of F' , this make U_t map F to F' along the flow. In particular, $g^{jk} = 0$ at the vertex, so U_t fix all vertex.

(4.2) is not a parabolic system, however we can use (3.2) to modify it, recall that

$$\sum_k g^{ik}(U) \frac{\partial U^j}{\partial y^k} = \sum_k g^{jk}(U) \frac{\partial U^i}{\partial y^k}, \text{ on } \bar{\mathcal{P}}$$

thus

$$\begin{aligned} \frac{\partial U^i}{\partial t} &= \frac{\partial}{\partial y^k} \left(g^{ij}(U) \frac{\partial U^k}{\partial y^j} \right) - (g^{ij})_l(U) \frac{\partial U^l}{\partial y^k} \frac{\partial U^k}{\partial y^j} \\ (4.3) \quad &= \frac{\partial}{\partial y^k} \left(g^{kj}(U) \frac{\partial U^i}{\partial y^j} \right) - (g^{ij})_l(U) \frac{\partial U^l}{\partial y^k} \frac{\partial U^k}{\partial y^j} \\ &= g^{kj}(U) \frac{\partial^2 U^i}{\partial y^k \partial y^j} - (g^{ij})_l(U) \frac{\partial U^l}{\partial y^k} \frac{\partial U^k}{\partial y^j} + (g^{kj})_l(U) \frac{\partial U^l}{\partial y^k} \frac{\partial U^i}{\partial y^j} \end{aligned}$$

where $(g^{ij})_l = \frac{\partial}{\partial y^l} g^{ij}$ is the derivatives of $g^{ij}(y)$.

(4.3) is a quasi-linear parabolic system degenerated on the boundary, the equation in second row have a nice divergence form. A direct computation show that if the solution of (4.3) satisfies (3.2) at $t = 0$ then will satisfy it all

the time. We already know that (4.3) has long time solution from the origin J-flow (1.2), the question is how U_t will behave as time tend to infinity.

4.1. Some Basic Estimates.

Lemma 4.1. $\inf_{t=0} \text{tr}_{\omega_\varphi} \alpha \leq \text{tr}_{\omega_\varphi} \alpha \leq \sup_{t=0} \text{tr}_{\omega_\varphi} \alpha.$

Proof. As in [3], take derivative of (1.2) w.r.t. time,

$$\frac{\partial^2 \varphi}{\partial t^2} = g_{\varphi}^{i\bar{l}} \alpha_{i\bar{j}} g_{\varphi}^{k\bar{j}} \left(\frac{\partial \varphi}{\partial t} \right)_{k\bar{l}}$$

then applied the maximal principle. We also can use (4.2), let $\delta > 0$, suppose $G \triangleq \text{tr} DU - \delta t$ at (y_0, t_0) take the maximum value over $\bar{\mathcal{P}} \times [0, T]$. If $t_0 > 0$, we consider the case when $y_0 \in F^\circ$, F is a 1-codimensional face, the other case is similar. Choose a coordinate such that $\mathcal{P} \subset \{y^n \geq 0\}$, $F = \mathcal{P} \cap \{y^n = 0\}$, then at (y_0, t_0)

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial t} G = \sum_{p < n} \left(\frac{\partial U^p}{\partial t} \right)_p + \left(\frac{\partial U^n}{\partial t} \right)_n - \delta \\ &= \sum_{p, q < n} g^{pq}(U) G_{pq} + \sum_{p, q < n} (g^{pq}(U))_p G_q \\ &\quad + (g^{nn})_n(U) \frac{\partial U^n}{\partial y^n} G_n - \delta \\ &\leq -\delta \end{aligned}$$

Note that $[g^{pq}(U(y_0))] > 0$, $g^{ni}(U(y_0)) = 0$, $(g^{nn})_n(U(y_0)) \geq 0$, $[G_{pq}(y_0)] \leq 0$, $G_q(y_0) = 0$, $G_n(y_0) \leq 0$, $\frac{\partial U^n}{\partial y^n}(y_0) > 0$. It is a contradiction, so $t_0 = 0$. Then let $\delta \rightarrow 0$, we get the upper bound, for the lower bound is similar. \square

The above estimate gives upper bound of the eigenvalues of DU . At a point, choose a basis $\{e_i\}$ of $T^{1,0}X$ such that $\langle e_i, e_j \rangle_{\omega_\varphi} = \delta_{ij}$, $\langle e_i, e_j \rangle_\alpha = \lambda_i \delta_{ij}$, $\lambda_i > 0$, $Ae_i = \lambda_i e_i$, A is the linear transform induced by ω_φ and α . From proposition 3.2, λ_i is also the eigenvalue of DU ,

$$(4.4) \quad \|A\|^2 \triangleq \text{tr}(AA^*) = \text{tr}(A^2) = \sum \lambda_i^2 < \left(\sum \lambda_i \right)^2 = (\text{tr} A)^2$$

where A^* is the dual transform w.r.t ω_φ or α . So $\|A\|^2$ is bounded uniformly along the flow, moreover on U_0 , with (2.8), (3.3)

$$\text{tr}(AA^*) = A_j^i \bar{A}_l^{\bar{k}} \Phi_{i\bar{k}} \Phi^{j\bar{l}} = A_j^i \bar{A}_l^{\bar{k}} F_{i\bar{k}} F^{j\bar{l}} = \frac{\partial U^j}{\partial y^i} \frac{\partial U^l}{\partial y^k} g^{ik}(U) g_{jl}(U)$$

Theorem 4.2. *The eigenvalues of DU_t are positive and upper bounded uniformly. In the interior of \mathcal{P} , we have a partial bound on DU ,*

$$(4.5) \quad \sum_{i,j,k,l} \frac{\partial U^j}{\partial y^i} \frac{\partial U^l}{\partial y^k} g^{ik}(U) g_{jl}(U) \leq C$$

and on $\bar{\mathcal{P}}$,

$$(4.6) \quad \det DU \leq C'$$

To get the version of (4.5) on $\partial\mathcal{P}$, take a point y in the interior of face F , $\dim F = p$. Since $DU|_y : M_R \rightarrow M_R$ have an invariant subspace TF , it induces $DU|_{M_R/TF} : M_R/TF \rightarrow M_R/TF$, because F is the intersection of $n - p$ $(p + 1)$ -dimensional face and U map face to face, so M_R/TF can be decomposed to a direct sum of 1-dimensional invariant subspace, and the eigenvalues on these 1-dim subspace are also eigenvalues of DU , so they are positive and bounded uniformly. For $DU|_{TF} : TF \rightarrow TF$, it is self-dual and positive w.r.t. the metric on TF , namely $D^2g|_{F'}$, F' is the corresponding face.

For example, $\dim F = n - 2$, choose a coordinate such that F is parallel to $\{y^n = y^{n-1} = 0\}$,

Corollary 4.3. *On the face F , the eigenvalues of DU are constituted of $\frac{\partial U^n}{\partial y^n}$, $\frac{\partial U^{n-1}}{\partial y^{n-1}}$ and eigenvalues of $DU|_{TF}$, they are positive and upper bounded uniformly, and in the interior of F ,*

$$\sum_{i,j,k,l \leq n-2} \frac{\partial U^j}{\partial y^i} \frac{\partial U^l}{\partial y^k} g^{ik}(U) g_{jl}(U) \leq C$$

Remark 4.4. The reason that we call (4.5) is just a partial bound is, for $y \in \mathcal{P}^\circ$, choose a basis of M_R such that $g_{ij}(U_t(y)) = \mu_i \delta_{ij}$, then (4.5) is $\sum_{i,j} \left(\frac{\partial U^j}{\partial y^i} \right)^2 \frac{\mu_j}{\mu_i} \leq C$, but when $t \rightarrow \infty$, $U_t(y)$ may approach to $\partial\mathcal{Q}$, there exists some i, j such that $\frac{\mu_j}{\mu_i}$ go to zero, so we can't bound $\frac{\partial U^j}{\partial y^i}(y)$ from (4.5). Take $i = j$, $\left| \frac{\partial U^i}{\partial y^i} \right|$ is bounded uniformly.

Theorem 4.5. *The flow converges to a smooth solution of (1.1) if and only if there exists $\delta > 0$ such that $\det DU \geq \delta$ uniformly.*

Proof. The necessity is trivial. Conversely if $\det DU \geq \delta$ uniformly, since the eigenvalues of DU is upper bounded uniformly, so is below bounded uniformly, hence $\text{tr}_\alpha \omega_\varphi$ is bounded uniformly from both side, then by the arguments in [17], flow converges to the solution of (1.1). \square

As the counterpart of Calabi's functional, we have the energy functional E ,

$$(4.7) \quad E_\alpha(\omega) = \frac{1}{2} \int_X (\text{tr}_\omega \alpha)^2 \frac{\omega^n}{n!} = \frac{1}{2} \int_{\mathcal{P}} (\text{tr} DU)^2 dy$$

Proposition 4.6. *Energy functional E is non-increasing along the flow.*

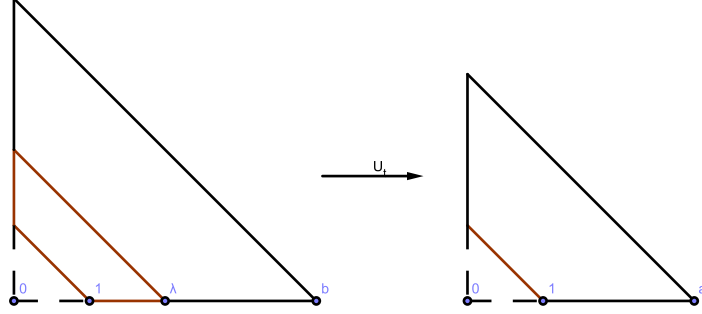


FIGURE 4.1.

Proof. by (4.2)

$$\begin{aligned}
 \frac{dE}{dt} &= \int_{\mathcal{P}} \text{tr} DU \frac{\partial}{\partial y^k} \left(\frac{\partial U^k}{\partial t} \right) dy \\
 &= \int_{\mathcal{P}} \text{div} \left(\text{tr} DU \cdot \frac{\partial U}{\partial t} \right) dy - \int_{\mathcal{P}} \frac{\partial \text{tr} DU}{\partial y^k} \frac{\partial U^k}{\partial t} dy \\
 (4.8) \quad &= - \int_{\mathcal{P}} g^{kl}(U) \frac{\partial \text{tr} DU}{\partial y^k} \frac{\partial \text{tr} DU}{\partial y^l} dy \leq 0
 \end{aligned}$$

since on the boundary of \mathcal{P} , $\frac{\partial U}{\partial t}$ is along the face, so the divergence term is zero. \square

Note that we also have

$$\frac{dE}{dt} = - \int_{\mathcal{P}} \frac{\partial \text{tr} DU}{\partial y^k} \frac{\partial U^k}{\partial t} dy = - \int_{\mathcal{P}} g_{kl}(U) \frac{\partial U^k}{\partial t} \frac{\partial U^l}{\partial t} dy \leq 0$$

By [3], we know that E has the same critical point as J-functional, namely the solution of (1.1). It is interesting to consider the variational problem

$$\min \left\{ \int_{\mathcal{P}} (\text{tr} DU)^2 dy \mid U : \bar{\mathcal{P}} \rightarrow \bar{\mathcal{Q}}, U \text{ s.t. (3.2)} \right\}$$

Example 4.7. In [7], Fang and Lai study the J-flow on the \mathbb{P}^n blow-up 1 point under the assumption of Calabi symmetry, namely the metrics are $U(n)$ -invariant, in our toric setting require metrics are $(\mathbb{S}^1)^n$ -invariant, to make their results fit into the toric setting, we need to require the symplectic potentials have more symmetry. Let

$$\mathcal{P} = \left\{ y \in \mathbb{R}^n \mid y^i \geq 0, 1 \leq \sum y^i \leq b \right\}$$

it gives the \mathbb{P}^n blow-up 1 point with Kähler class $[\omega] = b[E_\infty] - [E_0]$, E_∞ is the pull-back of hyperplane divisor by $X \rightarrow \mathbb{P}^n$, E_0 is the exceptional divisor.

Let $\mathcal{Q} = \{y \mid y^i \geq 0, 1 \leq \sum y^i \leq a\}$, it corresponds $[\alpha] = a[E_\infty] - [E_0]$. We require the symplectic potentials have the following form, denote $B = \sum y^i$

$$u = \sum y^i \log y^i - B \log B + h(B)$$

where h is a convex function defined on $[1, b]$ and satisfies $h(B) - (B - 1) \log(B - 1) - (b - B) \log(b - B)$ is smooth on $[1, b]$, then we can check that u satisfies Guillemin's conditions. In the same way, symplectic potential defined on \mathcal{Q} has form $g = \sum y^i \log y^i - B \log B + \theta(B)$, let the Legendre transform of h and θ is p and η respectively, they are defined on \mathbb{R} . Then the induced transition map U between polytopes is

$$U^i(y) = \eta'(h'(\sum y^i)) \frac{y^i}{\sum y^i}$$

Let $f(B) = \eta'(h'(B))$, then f is a smooth function that map $[1, b]$ to $[1, a]$, $0 < f' < \infty$.

Suppose $u(y, t)$ is a solution of J-flow (4.1), it preserves its form along the flow, h, p and f changes by time. The evolve equation (4.2) is reduced to

$$\frac{\partial f}{\partial t} = \frac{1}{\theta''(f)} \left(\frac{\partial^2 f}{\partial B^2} + (n-1) \frac{1}{B} \frac{\partial f}{\partial B} - (n-1) \frac{1}{B^2} f \right)$$

Note that $nc = n \frac{ab^{n-1}-1}{b^{n-1}-1}$. For the limit behavior of flow, there are three cases which be up to nc .

- Case 1.* $nc > n-1$, the flow converges to a smooth solution of (1.1). $U_t \rightarrow U_\infty$ is a diffeomorphism between polytopes satisfies $trDU_\infty \equiv nc$.
- Case 2.* $nc = n-1$, the flow converges to a metric with conic singularity along E_0 , and is a smooth solution of (1.1) on $X \setminus E_0$. $U_t \rightarrow U_\infty$ is a smooth one-to-one map between polytopes but not a diffeomorphism, $\det DU_\infty = 0$ on face $F_0 = \{y \mid \sum y^i \equiv 1\}$ which corresponds E_0 . $trDU_\infty \equiv nc$, and on F_0 , $trDU_\infty|_{TF_0} = trDU_\infty$.
- Case 3.* $nc < n-1$, the most interesting case, $\omega_t \rightarrow \omega_\infty + (\lambda - 1)[E_0]$ is a Kähler current, $[\omega_\infty] = b[E_\infty] - \lambda[E_0]$, where $\lambda \in (1, b)$ is determinate by

$$(n-1) \frac{b}{\lambda} + \frac{\lambda^{n-1}}{b^{n-1}} = na$$

Note that $nc' = n \frac{[\omega_\infty]^{n-1} \cup [\alpha]}{[\omega_\infty]^n} = n \frac{ab^{n-1}-\lambda^{n-1}}{b^{n-1}-\lambda^{n-1}} < nc$, the above equation is equivalent to $nc' = \frac{n-1}{\lambda}$.

ω_∞ is a metric with conic singularity along E_0 , and is a smooth solution of $c'\omega^n = \omega^n \wedge \alpha$ on $X \setminus E_0$.

$U_t \rightarrow U_\infty$ is just \mathcal{C}^1 map which squeeze the region $\{1 \leq \sum y^i \leq \lambda\}$ onto the face $\{y \mid \sum y^i \equiv 1\}$ of \mathcal{Q} , so on this region $\det DU_\infty = 0$, and $trDU_\infty = \frac{n-1}{\sum y^i}$. Its second derivative jump at $\{\sum y^i \equiv \lambda\}$.

U_∞ map $\{\lambda \leq \sum y^i \leq b\}$ onto \mathcal{Q} , in this region satisfies $trDU_\infty \equiv nc'$, and $\det DU_\infty = 0$ only on $\{\sum y^i \equiv \lambda\}$. In this case, we see

the gradient of symplectic potential ∇u_t will blow up in $\{1 \leq \sum y^i \leq \lambda\}$.

From the partial bound of derivative (4.5), we can prove the following property.

Proposition 4.8. *Suppose y_t is path of points in \mathcal{P}° , and there exists a domain $\Omega \subset \subset \mathcal{Q}$, such that $U_t(y_t) \in \Omega$ for $t > T$, then for any domain Ω_1 such that $\Omega \subset \subset \Omega_1 \subset \subset \mathcal{Q}$, there exists $\epsilon > 0$, such that $B_\epsilon(y_t) \subset \mathcal{P}$, and $U_t(B_\epsilon(y_t)) \subset \Omega_1$ for $t > T$. $B_\epsilon(y_t)$ is the Euclidean ball, the distance and length in the proof is w.r.t. the Euclidean metric.*

Proof. assume $d(\bar{\Omega}, \partial\Omega_1) > \delta$, there exists a point $b \in \partial U_t^{-1}(\Omega_1)$ such that $d(y_t, b) = d(y_t, \partial U_t^{-1}(\Omega_1))$, let l be the segment located in $U_t^{-1}(\bar{\Omega}_1)$ connected y_t and b , since $U_t(U_t^{-1}(\bar{\Omega}_1)) = \bar{\Omega}_1 \subset \subset \mathcal{P}$, by (4.5) we know on $U_t^{-1}(\bar{\Omega}_1)$ the derivatives $\sum \left(\frac{\partial U^j}{\partial y^i} \right)^2 \leq C$ for all time, so the length of curve $\mathbf{L}(U_t(l)) \leq C' d(y_t, b)$, and $U_t(l)$ connect $U_t(y_t) \in \Omega$ and $U_t(b) \in \partial\Omega_1$, so $\mathbf{L}(U_t(l)) \geq \delta$. Thus $d(y_t, \partial U_t^{-1}(\Omega_1)) = d(y_t, b) \geq \delta/C'$, then $B_{\delta/2C'}(y_t) \subset U_t^{-1}(\Omega_1) \subset \mathcal{P}$ for $t > T$. \square

Remark 4.9. By this property, we take a point $z \in \mathcal{Q}^\circ$, the inverse image $U_t^{-1}(z) = y_t$, then $d(y_t, \partial\mathcal{P}) > \epsilon$. In particularly, the distance from the minimum point of u_t to $\partial\mathcal{P}$ has a uniform lower bound.

Finally, we make some speculation. First, if we can prove that $y_t \rightarrow y_\infty \in \mathcal{P}^\circ$, then $U_t(y_\infty) \rightarrow z$, by the above proposition, there exists $\delta > 0$ such that $B_\delta(y_\infty) \subset \mathcal{P}^\circ$ and $d(U_t(B_\delta(y_\infty)), \partial\mathcal{Q}) > c$, then on $B_\delta(y_\infty)$, we have uniform derivative bound by (4.5) and (4.3) is strictly parabolic, we may show U_t converges on this ball. Then union these balls together we get a open set $\Theta \subset \mathcal{P}$, $U_t \rightarrow U_\infty$ on Θ , and $U_\infty(\Theta) = \mathcal{Q}$ for the arbitrariness of z . This means that U_t finally squeeze $\mathcal{P} \setminus \Theta$ onto $\partial\mathcal{Q}$, as the case 3 in the example.

Acknowledgements. I would like to thank my advisor Gang Tian for constant encouragement, Yalong Shi for many times useful discussion and Jiaqiang Mei for his help.

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